Welcome to Math 31CH: Honors Vector Calculus

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→ teaching
→ this course

↓

will contain: Syllabus
Exam Schedule
HW
Office Hours
TA Information

Book

Vector Calculus, Linear Algebra, and Differential Forms (4th Ed)
by Hubbard & Hubbard.

Grading Scheme: HW, 2 Exams, Final

	HW	Exam	Exam	Final
	20%	20%	20%	40%
or	20%,	20% (Highest exam)		60%

(Whichever is higher)

Goals: Integration (Multivariable)
Differential Forms
Exterior Derivatives
Generalized Stokes Theorem

Recall the Fundamental Theorem of Calculus

If f is continuous and F is an antiderivative of f then

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

FTOC

Writing $\frac{dF}{dt} = f$, we can rewrite the FTOC as:

$$\int_{[a,b]} f(t)\,dt = F(b) - F(a)$$

$$\int_{[a,b]} dF = \int_{\{a\}\cup\{b\}} F$$

$[a,b]$: call this interval Ω

$\{a\}\cup\{b\}$: This is the boundary of Ω: $\partial\Omega$

← don't worry about making this rigorous now (we'll do it later)

In words: The integral of dF over the oriented interval $[a,b]$ is the same as the integral of F over the oriented boundary of the interval.

So, we can rewrite the FTOC as

$$\int_{\Omega} dF = \int_{\partial\Omega} F$$

Generalized Stoke's Theorem:

Let Ω be a compact piece-with boundary of a k-dimensional oriented Manifold $M \subset \mathbb{R}^n$. Give the boundary $\partial\Omega$ the boundary orientation, and let F be a $(k-1)$-form defined on an open set containing Ω. Then

$$\int_{\partial\Omega} F = \int_{\Omega} dF$$

We will spend a large portion of the course building up to understand this Theorem.

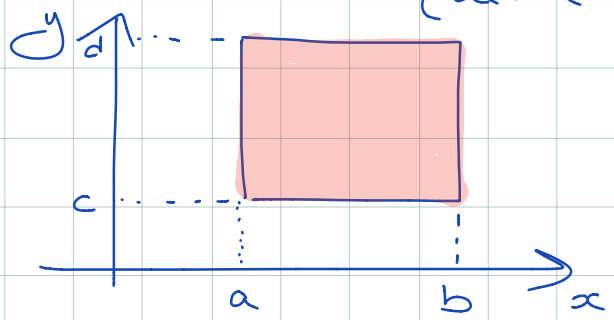
This theorem (and its special cases in 2d & 3d) have important applications in Physics

example: Maxwell's equations

Let's Get Started With Some Review

Chapter 4.1: n-dimensional Integrals

Example: Want to compute the total mass of rectangular plate $(a \le x \le b, c \le y \le d)$, with density $\rho(x, y)$



$$\iint_{[a,b]\times[c,d]} \rho(x,y)\, dx\, dy$$

What if ρ was a function of n variables, and the region was more complicated?

Riemann Integral

Suppose $f: \mathbb{R}^n \to \mathbb{R}$ is bounded and has bounded support

Recall: Dyadic Cube $C_{\vec{k},N} \subset \mathbb{R}^n$ is given by

($\vec{k}$: location of bottom left corner of cube; N: "level")

$$C_{\vec{k},N} = \left\{ \vec{x} \in \mathbb{R}^n \;\middle|\; \frac{k_i}{2^N} \leq x_i < \frac{k_{i+1}}{2^N}, \; 1 \leq i \leq n \right\}$$

$\vec{k} = (k_1, k_2, \ldots, k_n)$

Example: $C_{(3,7),4} = \left\{ x \in \mathbb{R}^2 \;\middle|\; \frac{3}{2^4} \leq x < \frac{4}{2^4}, \; \frac{7}{2^4} \leq y < \frac{8}{2^4} \right\}$

N^{th} Dyadic Paving: $D_N(\mathbb{R}^n)$ = collection of cubes $C_{\vec{k},N}$ at level N

Upper & Lower Sums

$$U_N(f) = \underbrace{\frac{1}{2^{nN}}}_{\text{volume of a cube}} \sum_{C \in D_N} \underbrace{M_C(f)}_{\text{least upper bound of } f \text{ over } C}$$

(side $= 2^{-N}$) $\Rightarrow$ vol $= (2^{-N})^n$

$$L_N(f) = \frac{1}{2^{nN}} \sum_{C \in D_N} \underbrace{m_C(f)}_{\text{greatest lower bound of } f \text{ over } C}$$

Upper & Lower Integrals

$$U(f) = \lim_{N \to \infty} U_N(f) \qquad L(f) = \lim_{N \to \infty} L_N(f)$$

Definition: $f: \mathbb{R}^n \to \mathbb{R}$ is integrable if $U(f) = L(f)$.

We then denote its integral by

$$\int_{\mathbb{R}^n} f \, |d^n \vec{x}| = U(f) = L(f)$$

Properties: Recall that

- $\int_{\mathbb{R}^n} (f+g)\,|d^n\vec{x}| = \int_{\mathbb{R}^n} f\,|d^n\vec{x}| + \int_{\mathbb{R}^n} g\,|d^n\vec{x}|$
- $\int_{\mathbb{R}^n} af\,|d^n\vec{x}| = a\int_{\mathbb{R}^n} f\,|d^n\vec{x}|$ ($a \in \mathbb{R}$)
- $f \leq g \Rightarrow \int_{\mathbb{R}^n} f\,|d^n\vec{x}| \leq \int_{\mathbb{R}^n} g\,|d^n\vec{x}|$
- $\left|\int_{\mathbb{R}^n} f\,|d^n\vec{x}|\right| \leq \int_{\mathbb{R}^n} |f|\,|d^n\vec{x}|$

Volume:

when $\mathbb{1}_A$ (indicator) is integrable

$$\text{vol}_n(A) = \int_{\mathbb{R}^n} \mathbb{1}_A\,|d^n\vec{x}|$$

Parable Set: A is parable if $\mathbb{1}_A$ is integrable

Exercise: Let $A = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$

Show that $\text{Vol}_n(A) = \prod_{i=1}^{n} |b_i - a_i|$

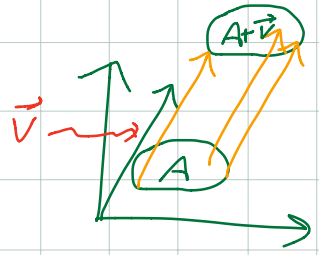
Properties of Volume:

- $A \cap B = \emptyset$, A & B are paraable

$$\Rightarrow \text{Vol}_n(A \cup B) = \text{Vol}_n(A) + \text{Vol}_n(B)$$

- $\text{Vol}_n(\underbrace{A + \vec{v}}) = \text{Vol}_n(A)$

$= \{\vec{x} + \vec{v} \mid \vec{x} \in A\}$



- $\text{Vol}_n(\underbrace{tA}) = t^n \text{Vol}_n(A)$

$\hookrightarrow \{t\vec{x} \mid \vec{x} \in A\}$

- $\text{Vol}_n(X) = 0 \iff \forall \epsilon > 0, \exists N$ s.t.

$$\sum_{\substack{C \in \mathcal{D}_N(\mathbb{R}^n) \\ C \cap X \neq \emptyset}} \text{Vol}_n(C) \leq \epsilon$$

Can you think of sets in $\mathbb{R}^3$ with volume $= 0$?

Section 4.5 : Fubini's Theorem & Iterated Integrals (Review)

Let f be an integrable function on $\mathbb{R}^n \times \mathbb{R}^m$ and suppose that for each $\vec{x} \in \mathbb{R}^n$, the function $\vec{y} \longmapsto f(\vec{x},\vec{y})$ is integrable.

Then, the function

$$\vec{x} \longmapsto \int_{\mathbb{R}^m} f(\vec{x},\vec{y})\,|d^m\vec{y}|$$

is integrable, and

$$\int_{\mathbb{R}^{m+n}} f(\vec{x},\vec{y})\,|d^n\vec{x}||d^m\vec{y}| = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(\vec{x},\vec{y})\,|d^m\vec{y}| \right) |d^n\vec{x}|$$

Fubini's Thm

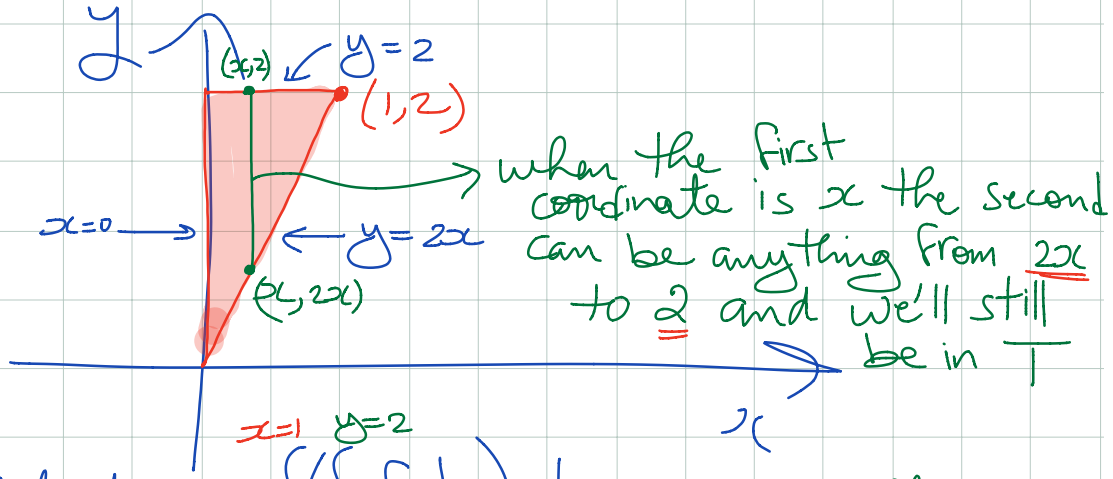
In words: You can write a multiple integral as an iterated integral "as long as everything you try to integrate is integrable" :o =

Example: Compute the integral of

$f(x,y) = x^2 + y^3$ over the region

$T = \{(x,y) \in \mathbb{R}^2 \mid 2x \le y, \text{ where } 0 \le y \le 2\}$

Sol'n: Let's start by drawing a picture



$$\iint f\,dx\,dy = \int_{x=0}^{x=1}\left(\int_{y=2x}^{y=2} f\,dy\right)dx$$

we could evaluate either one of these two Let's do the second

$$= \int_{y=0}^{y=2}\left(\int_{x=0}^{x=y/2} f\,dx\right)dy$$

$$= \int_0^2 \int_0^{y/2} x^2 + y^3\,dx\,dy$$

$$= \int_0^2 \left[\frac{x^3}{3} + xy^3\right]_0^{y/2} dy = \int_0^2 \frac{y^3}{24} + \frac{y^4}{2}\,dy$$

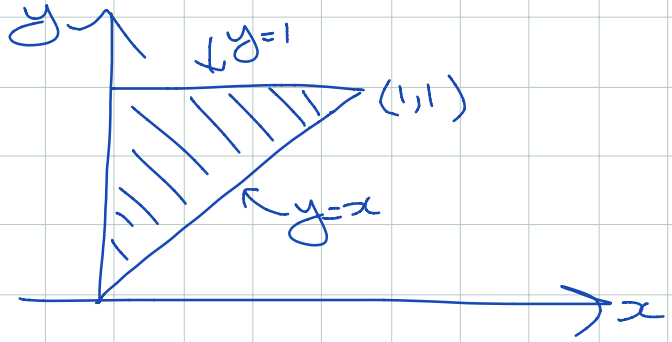
$$= \frac{y^4}{96} + \frac{y^5}{10}\Big|_0^2 = \frac{1}{6} + \frac{32}{10} = \cdots$$

Fubini's Thm can be very useful because we often don't know how to evaluate an integral in one order but it's easy in the other.

Example: Consider the iterated integral

$$\int_0^1 \int_x^1 e^{y^2}\, dy\, dx$$

Here, e^{y^2} doesn't have an antiderivative that we know, so we try changing the order



$$\int_0^1 \int_x^1 e^{y^2}\, dy\, dx = \int_0^1 \int_0^y e^{y^2}\, dx\, dy$$

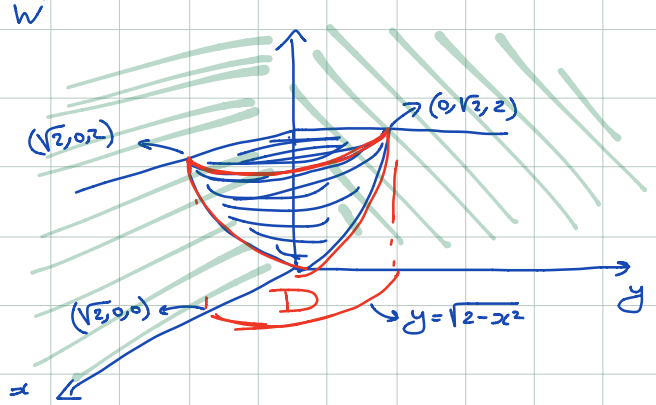
$$= \int_0^1 x e^{y^2}\Big|_{x=0}^{x=y} dy = \int_0^1 y e^{y^2} dy$$

$$= \frac{1}{2} e^{y^2}\Big|_{y=0}^{y=1} = \frac{e^1 - e^0}{2} = \frac{e-1}{2}$$

Example: Let W be the region bounded by $x=0$, $y=0$, $z=2$, & the surface $z=x^2+y^2$ lying in the quadrant $x\geq 0$, $y\geq 0$.

(the planes)

Compute $\iiint_W x\,dx\,dy\,dz$



So, we can say $x^2+y^2 \leq z \leq 2$, $0\leq y\leq \sqrt{2-x^2}$, $0\leq x\leq \sqrt{2}$

So, $$\iiint_W x\,dV = \int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\int_{x^2+y^2}^{2} x\,dz\,dy\,dx$$

$$= \int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}} 2x - x^3 - y^2x\,dy\,dx$$

$$= \int_0^{\sqrt{2}} 2xy - x^3y - y^3/3\,x \Big|_{y=0}^{y=\sqrt{2-x^2}} dx$$

$$= \int_0^{\sqrt{2}} 2x\sqrt{2-x^2} - x^3\sqrt{2-x^2} - \frac{(2-x^2)^{3/2}}{3}x\,dx$$

$$= \int_0^{\sqrt{2}} x(2-x^2)^{3/2} - x(2-x^2)^{3/2}/3 \;\; dx$$

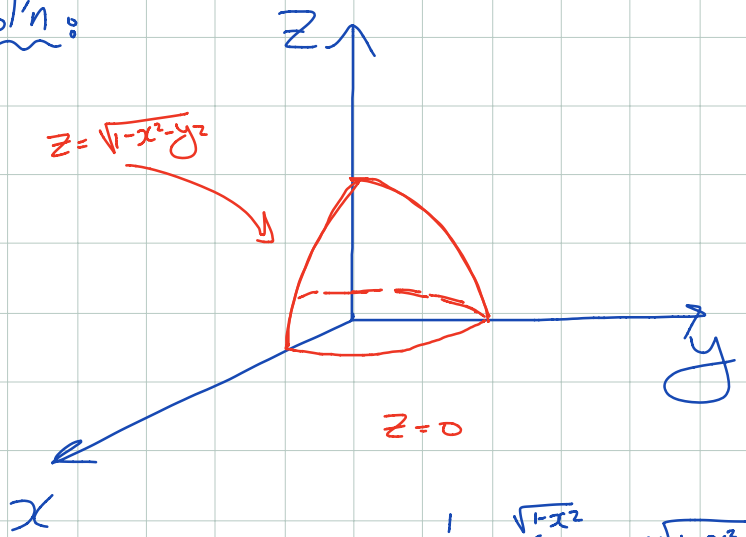
$$= \int_0^{\sqrt{2}} \frac{2}{5} x(2-x^2)^{3/2}\, dx$$

$$\underset{x:0\to\sqrt{2}}{=} \int \frac{1}{3}(2-x^2)^{3/2}\, dx^2$$

$$\underset{u=2-x^2}{=} \int_2^0 -\frac{1}{3} u^{3/2}\, du = \frac{2}{5}\cdot\frac{1}{3} 2^{5/2} = 8\sqrt{2}/15$$

Example: Set up the triple integral for finding the Volume & Mass of an object W bounded by the planes $z=0$, $y=0$ & the graph of $z=\sqrt{1-x^2-y^2}$, with density $f(x,y,z)$.

Soln:



So Volume $= \int_{x=-1}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} dz\, dy\, dx$

Mass $= \int_{x=-1}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} f(x,y,z)\, dz\, dy\, dx$

Example (part 2) Find the Mass if $f(x,y,z) = y$

$$M = \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} y \, dz\, dy\, dx$$

$$= \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\, y \, dy\, dx \qquad \text{Let } y^2 = u \Rightarrow 2y\,dy = u$$

$$= \int_{-1}^{1}\int_{y=0}^{\sqrt{1-x^2}} \frac{1}{2}\sqrt{1-x^2-u}\, du\, dx \qquad \text{Let } V = 1-x^2-u,\ dV = -du$$

$$= \int_{-1}^{1}\int_{y=0}^{\sqrt{1-x^2}} -\frac{1}{2}\sqrt{\underbrace{1-x^2-u}_{V}}\, d(\overbrace{1-x^2-u}^{V})\, du$$

$$= \int_{-1}^{1} -\frac{1}{2}\cdot\frac{2}{3}\left(1-x^2-u\right)^{3/2}\Big|_{y=0}^{\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} -\frac{1}{3}\left(1-x^2-y^2\right)^{3/2}\Big|_{y=0}^{\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} -\frac{1}{3}\left[\underbrace{\left(1-x^2-(1-x^2)\right)^{3/2}}_{0} - \left(1-x^2-0\right)^{3/2}\right] dx$$

$$= \int_{-1}^{1} +\frac{1}{3}\left(1-x^2\right)^{3/2} dx = \frac{\pi}{8}$$

how?

Let $x = \sin u \Rightarrow dx = \cos u\, du$

& $1-x^2 = \cos^2 u$

so $\int_{x=-1}^{1} \frac{1}{3}\left[\cos^2(u)\right]^{3/2}\cos u\, du = \int_{x=-1}^{1} \cos^4 u\, du$

$= \ldots$ big parts

Aaaaah!

Turns out many problems (like this one) are much easier with a change of variables.

$\Rightarrow$ Next Couple of Sections

4.8 Determinants: (as you probably "know", we need this to calculate volumes of parallelograms)

Theorem

$\exists$ A unique function $D: (\mathbb{R}^n)^n \to \mathbb{R}$ such that (take n vectors in $\mathbb{R}^n$ to $\mathbb{R}$)

(1) D is linear with respect to each of its arguments (multilinearity)

(2) Interchanging any two arguments changes its sign (antisymmetry)

(3) $D(\vec{e}_1, \vec{e}_2, \dots \vec{e}_n) = 1$ (normalization)

Given an $n \times n$ matrix $A = [\vec{a}_1, \vec{a}_2, \dots \vec{a}_n]$

$\det(A) = D(\vec{a}_1, -, \vec{a}_n)$.

Def'n of determinant

- Proof of existence (see A.19)
- Proof of uniqueness

Step I: we need some properties of the determinant

(1) Suppose $A_1 = [\vec{a}_1, \dots, \vec{a}_j, \dots, \vec{a}_n]$

& $A_2 = [\vec{a}_1, \dots, \mu\vec{a}_j, \dots \vec{a}_n]$

then $D(A_2) = \mu D(A_1)$ by multilinearity

(2) Suppose $A_2 = [\vec{a}_1, \dots, \vec{a}_i, \dots, \vec{a}_j + \beta\vec{a}_i, \dots \vec{a}_n]$

then $D(A_2) = D(A_1)$

bec $D(A_2) =$ $D(A_1) + \beta D([\vec{a}_1, \dots, \vec{a}_i, \dots, \vec{a}_i, \dots \vec{a}_n])$

by multilinearity

$= 0$ by HW exercise 4.8.7

Step II: Suppose Δ' is another function that satisfies the properties in the theorem. we want to show that $\det(A) = \Delta'(A)$.

Recall that we can column reduce a square matrix till we end up with the identity matrix I or the zero matrix.

Let's write this as

$$A \xrightarrow{\mu_1} A_1 \xrightarrow{\mu_2} A_2 \xrightarrow{\mu_3} \cdots \xrightarrow{\mu_{p-1}} A_{p-1} \xrightarrow{\mu_p} A_p$$

If $A_p = 0$, then by property (1) $\Delta(A_p) = \Delta'(A_p) = 0$
(HW 4.8.7)

If $A_p = I$, then since $\Delta(I) = 1$, $\Delta(A_p) = 1 = \Delta'(A_p)$
& $\Delta'(I) = 1$

$$\Rightarrow \Delta(A_{p-1}) = \Delta(A_p)/\mu_p \quad \text{(by step I)}$$
$$= \Delta'(A_p)/\mu_p \quad \text{(by the green cloud)}$$
$$= \Delta'(A_{p-1}) \quad \text{(by step I)}$$

Similarly $\Delta(A_{p-2}) = \Delta'(A_{p-2})$

$$\vdots$$

$$\Delta(A) = \frac{1}{\mu_1\mu_2\cdots\mu_p}\Delta(A_p) = \frac{1}{\mu_1\mu_2\cdots\mu_p}\Delta'(A_p) = \Delta'(A)$$

$\Rightarrow \Delta$ is unique! ∎

Remark: This also tells us that we can compute determinants by column reduction

$$\det(A) = \frac{1}{\mu_1 \mu_2 \cdots \mu_p} \det(A_p) = \begin{cases} \frac{1}{\mu_1 \cdots \mu_p} & \text{if } A_p = I \\ 0 & \text{if } A_p = 0 \end{cases}$$

Theorem:

A is invertible if & only if $\det(A) \neq 0$

Proof: We know A is invertible if & only if it can be column reduced to the identity.

Theorem: If A & B are $n \times n$ matrices

$$\det(A)\det(B) = \det(AB)$$

Proof: There are two cases: either A is invertible or not

Case I: A is not invertible

$\Rightarrow \det(A) = 0$ & $\text{rank}(A) < n$

but $\text{img}(AB) \subset \text{img}(A) \Rightarrow \text{rank}(AB) \leq \text{rank}(A) < n$

$\Rightarrow AB$ is not invertible $\Rightarrow \det(AB) = 0$

Case 2 A is invertible

Consider $F(B) = \dfrac{\det(AB)}{\det(A)}$

we would be done if $F(B) = \det(B)$
so we just have to show that $F(B)$ satisfies properties 1, 2, 3 of the definition of determinants

Exercise 4.8.8 (HW)

very easy

☺

If A is invertible

Corollary: $\det(A^{-1}) = \frac{1}{\det(A)}$

Proof: $\det(\underbrace{AA^{-1}}_{I}) = \det(A)\det(A^{-1})$

$\det(I) = 1$

Corollary: If P is invertible

$$\det(A) = \det(P^{-1}AP)$$

Proof: $\det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P)$

$= \det(P^{-1})\det(P)\det(A)$

determinants are just numbers

$= \det(P^{-1}P)\det(A)$

$= \det(A)$

∎

Theorem: For any $n \times n$ matrix,

$$\det A = \det A^T$$

Proof:

(I) Recall the three types of elementary matrices:

$E_1(i,x)$ ⇝ diagonal matrix with ones on the diagonal, except the $(i,i)^{th}$ entry (which is x)

$E_2(i,j,x)$ ⇝ diagonal matrix with ones on the diagonal, except that the $(i,j)^{th}$ entry is x

$E_3(i,j)$ ⇝ diagonal matrix with ones on the diagonal, except that the (i,i) & (j,j) entries are zero & the (i,j) & (j,i) entries are 1.

We will first prove the theorem for the elementary matrices

(1) $E_1(i,m)^T = E_1(i,m)$, so ✓

(2) $E_2(i,j,x)^T = E_2(j,i,x)$, so ✓

Rec $\det E_2(i,j,x) = 1$ (why?)
& $\det E_2(j,i,x) = 1$

(3) $E_3(i,j)^T = E_3(i,j)$, so ✓

$\det E_1(i,m) = m$ (multilinearity)

$\det E_2(i,j,x) = 1$ (why?)

$\det E_3(i,j) = -1$ (Exchanging two columns)

side note

(II) Recall that there are elementary matrices $E_1, \dots, E_k$, so that if we reduce A to its row echelon form $\tilde{A}$, we have

$$\tilde{A} = E_k \cdots E_1 A \quad \text{—(1)}$$

$$\Rightarrow \tilde{A}^T = A^T E_1^T E_2^T \cdots E_k^T \quad \text{— (2)}$$

$\tilde{A}^T$: column echelon form

(1) $\Rightarrow \det(\tilde{A}) = \det(E_k) \cdots \det(E_1) \det(A)$

(2) $\Rightarrow \det(\tilde{A}^T) = \det(E_k) \cdots \det(E_1) \det(A^T)$

So, either $\tilde{A} = I \Rightarrow \tilde{A}^T = \tilde{A} = I \Rightarrow \det(A) = \det(A^T)$

or $\tilde{A} \neq I \Rightarrow A$ is rank deficient & A^T too

$\Rightarrow \det(A) = \det(A^T) = 0$



Computing the determinant

• Given an $n \times n$ matrix A, denote by $A_{i,j}$ the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row & j^{th} column of A.

Then, the determinant is $\Delta_n : Mat(n,n) \longrightarrow \mathbb{R}$

where $\Delta_1([a]) = a$

& $$\Delta_n(A) = \sum_{i=1}^{n} (-1)^{1+i} a_{i,1} \Delta_{n-1}(A_{i,1}), \quad n > 1$$

↑ this is called the development according to the first column

Proof: A.19

Remark: since $\det A = \det A^T$ we could have "developed according to the first row".

Ex: $$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$\det(A) = 1 \cdot \det\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} - 0 \cdot \det\begin{bmatrix} 3 & 4 \\ 2 & 0 \end{bmatrix} + 1 \cdot \det\begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}$$

$$= 1 \cdot (0-2) - 0 \cdot (0-8) + 1 \cdot (3-4) = -3$$

Theorem: Let A be an $n \times n$ triangular matrix then $\det A = \prod_{i=1}^{n} a_{i,i}$

Proof: Enough to work with upper $\triangle$ (why?)

Induction:

Base case: $\det([a]) = a \quad (n=1)$

Hypothesis: $\det A_{1,1} = \prod_{i=2}^{n} a_{i,i}$

Now,

$$\det(A) = \sum_{i=1}^{n} (-1)^{1+i} a_{i,1} \triangle_{n-1}(A_{i,1}), \quad n>1$$

A is $\triangle$ $\longrightarrow$ $= (-1)^2 a_{1,1} \det(A_{1,1})$

$$= \prod_{i=1}^{n} a_{i,i}$$



Theorem 3:

$$\det\left(\left[\begin{array}{c:c} A & C \\ \hdashline 0 & B \end{array}\right]\right) = \det(A)\det(B)$$

(A: $n \times n$, C: $n \times m$, 0: $m \times n$, B: $m \times m$)

Proof: HW (4.8.11)

Permutations & their Signatures

Def'n: A permutation of a (finite) set X is a bijection $f: X \to X$. The set of all permutations of the set $\{1, 2, \ldots, n\}$ is denoted by Perm_n. A permutation has the following properties:

These properties essentially define groups

(1) If $\sigma_1, \sigma_2, \sigma_3 \in \text{Perm}_n$ then $(\sigma_1 \circ \sigma_2) \circ \sigma_3 = \sigma_1 \circ (\sigma_2 \circ \sigma_3)$

(associativity of composition)

(2) There is an identity for composition (i.e. $\exists\, id \in \text{Perm}_n$, s.t. $\sigma \circ I = \sigma$ $I \circ \sigma = \sigma$ $\forall \sigma \in \text{Perm}_n$)

(3) $\forall \sigma \in \text{Perm}_n, \exists\, \sigma^{-1} \in \text{Perm}_n$ s.t. $\sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = I$

Example: $\sigma \in \text{Perm}_3$ with $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$

Example: $\sigma \in Perm_5$ with

$$\sigma(1)=3,\ \sigma(2)=4,\ \sigma(3)=5,\ \sigma(4)=2,\ \sigma(5)=1$$

so: $1 \to 3 \to 5 \to 1$ ← this is a cycle
& $2 \to 4 \to 2$ ← this is a cycle

So, we can represent σ more concisely as

$$(1,3,5)(2,4)$$

Transposition: A cycle that exchanges only 2 elements & leaves all else unchanged

e.g. $(1)(2,3)(4)(5)$
$(1)(2)(3,5)(4)$

Signature: $\exists$ a unique map

$$sgn : Perm_n \to \{-1, 1\}$$

s.t. (1) $sgn(\sigma_1 \circ \sigma_2) = sgn(\sigma_1)\, sgn(\sigma_2)$ $\forall \sigma_1, \sigma_2 \in Perm_n$

(2) $sgn(\tau) = -1$ $\forall$ transposition $\tau \in Perm_n$

Proof: Existance

$\forall \sigma \in Perm_n$ Define the permutation matrix M_σ by $M_\sigma \vec{e}_i = \vec{e}_{\sigma(i)}$

then $\forall \sigma, \tau \in Perm_n \quad M_{\sigma\circ\tau} = M_\sigma M_\tau$

(because $\forall i$: $M_{\sigma\circ\tau}\vec{e}_i = \vec{e}_{(\sigma\circ\tau)(i)} = \vec{e}_{\sigma(\tau(i))} = M_\sigma \vec{e}_{\tau(i)} = M_\sigma M_\tau \vec{e}_i$)

Now, simply let $\operatorname{sgn}(\sigma) = \det(M_\sigma)$

$\Downarrow$

(1) $\operatorname{sgn}(\sigma\circ\tau) = \det(M_{\sigma\circ\tau}) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)$

(2) $\operatorname{sgn}(\tau) = -1$ (why?)

Uniqueness: Exercise

Theorem: $\det A = \sum_{\sigma \in Perm_n} \operatorname{sgn}(\sigma)\, a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$

We'll do an example, then prove it.

Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $(n=2)$

$\sigma_1 = (1,2)$ $\quad \operatorname{sgn}(\sigma_1) = -1$

$\sigma_2 = (1)(2)$ $\quad \operatorname{sgn}(\sigma_2) = +1$

$$\Rightarrow \det(A) = -1 \times (2)(3) + 1 \times (1)(4)$$

$$= -2$$

((2): $a_{1,\sigma_1(1)} = a_{1,2}$; (3): $a_{2,\sigma_1(2)} = a_{2,1}$; (1): $a_{1,1}$; (4): $a_{2,2}$)

Proof: We want to show that

$$D(A) := \sum_{\sigma \in \text{Perm}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

is the determinant. So we'll just check multilinearity, antisymmetry & normalization

multilinearity: Fixing all columns except the ith fixes all $a_{j,\sigma(j)}$ except the ith. So each term $\prod_{i=1}^{n} a_{i,\sigma_i}$ is multilinear and so their sum is too.

antisymmetry: see below

normalization: $D(I) = 1$ (easy)

$\exists i,\ a_{i,\sigma(i)} = 0$ unless $M_\sigma = I$

Antisymmetry: let A' be the matrix obtained by exchanging the ith & j^{th} columns of A

& let τ be the corresponding permutation

$$D(A') = \sum_{\sigma \in Perm_n} sgn(\sigma) \prod_{i=1}^{n} a'_{i,\sigma(i)}$$

$$= \sum_{\sigma \in Perm_n} sgn(\sigma) \prod_{i=1}^{n} a_{i,(\tau\circ\sigma)(i)}$$

(let $\sigma' = \tau \circ \sigma$ & reindex)

$$= \sum_{\sigma' \in Perm_n} sgn(\tau^{-1}\circ\sigma') \prod_{i=1}^{n} a_{i,\sigma'(i)}$$

$$= \sum_{\sigma' \in Perm_n} \underbrace{sgn(\tau^{-1})}_{-1} sgn(\sigma') \prod_{i=1}^{n} a_{i,\sigma'(i)}$$

$$= -D(A)$$



$n \times n$

Trace: The trace of a matrix is the sum of its diagonal elements

$$Tr(A) = \sum_{i=1}^{n} a_{ii}$$

Trace is linear

Theorem: The determinant function $det : mat(n,n) \to \mathbb{R}$ is differentiable & $[D\,det(A)]B = det(A)\,tr(A^{-1}B)$

Proof: See book

Example: 2×2 case, you can think of the det as a function f from $\mathbb{R}^4 \to \mathbb{R}$

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = f\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = ad - bc$$

with $\frac{\partial f}{\partial a} = d$, $\frac{\partial f}{\partial b} = -c$, $\frac{\partial f}{\partial c} = -b$, $\frac{\partial f}{\partial d} = a$

$$\Rightarrow \left[D f\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}\right] = [d, -c, -b, a]$$

$$\Rightarrow \left[D\det \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A}\right] \underbrace{\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}}_{B} = [d, -c, -b, a] \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}$$

$$= \alpha d - \beta c - \gamma b + \delta a$$

On the other hand • $\det(A) = ad - bc$

• $tr(A^{-1}B) = \dfrac{tr\left(\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right)}{ad - bc}$

$$tr(A^{-1}B) = tr\begin{pmatrix} \alpha d - \beta\gamma & * \\ * & -c\beta + a\delta \end{pmatrix} / ad - bc = \frac{\alpha d - b\gamma - c\beta + a\delta}{\det(A)}$$

and

$$[D\det(A)]B = \det(A)\,\mathrm{tr}(A^{-1}B)$$

Corollary:

$$\mathrm{tr}(P^{-1}AP) = \mathrm{tr}(A) \quad \forall \text{ invertible } P$$

Proof: see book

Defn: The characteristic polynomial of a square matrix A is $\chi_A(t) = \det(tI - A)$

Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\chi_A(t) = \det\begin{bmatrix} t-1 & -2 \\ -3 & t-4 \end{bmatrix} = (t-1)(t-4) - (-2)(-3)$$

Theorem: The eigenvalues of A are the roots of χ_A

Proof: (1) λ is a root of $\chi_A \Rightarrow \det(\lambda I - A) = 0$

$\Rightarrow \lambda I - A$ has a non-trivial null-space

If $\vec{v} \in \ker(\lambda I - A)$
then $\lambda \vec{v} = A\vec{v}$ so λ is an eigenvalue

(2) If $A\vec{v} = \lambda \vec{v}$ for some $\vec{v} \neq 0$
then $\vec{v} \in \ker(\lambda I - A)$
$\Rightarrow \det(\lambda I - A) = 0$

Corollaries: (1) The diagonal entries of a triangular matrix are its eigenvalues

can you prove it?

(2) For a matrix A, if the roots of χ_A are simple, then $\mathbb{C}^n$ admits an eigenbasis for A.

Cayley-Hamilton Theorem

If A is a square matrix then

$$\chi_A(A) = [0]$$

Example: we saw that if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\chi_A(t) = (t-1)(t-4) - 6$$

$$\Rightarrow \chi_A(A) = (A - I)(A - 4I) - 6I$$

$$= A^2 - A - 4A + 4I - 6I$$

$$= A^2 - 5A - 2I$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$